Closing Thu: TN 3
Closing Tue: TN 4 (sigma notation practice)
Closing next Thu: TN 5 (last assignment)
Final: Sat, June 2nd, 5:00-7:50pm, KANE 130

Taylor's Inequality (error bound):
On a given interval [a,b],
if $|f^{(n+1)}(x)| \leq M$, then

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-b|^{n+1}$$

Recall:

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(b)(x-b)^k$$

$$= \frac{1}{0!} f(b) + \frac{1}{1!} f'(b)(x-b) + \frac{1}{2!} f''(b)(x-b)^2 + \cdots + \frac{1}{n!} f^{(n)}(b)(x-b)^n$$

Entry Task:
Find the 7th Taylor polynomial for $f(x) = \sin(x)$, based at $b = 0$.
Find a bound on the error over the interval [-3,3].

TN 4: Taylor Series

Def'n:

The **Taylor Series** for f(x) based at b is

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(b)(x-b)^k = \lim_{n\to\infty} T_n(x)$$

If the limit exists at a particular *x*, then we say the series **converges** at *x*. Otherwise, we say it **diverges** at x.

The **open interval of convergence** is the largest open interval of values over which the series converges.

Note:

If

$$\lim_{n\to\infty} \frac{M}{(n+1)!} |x-b|^{n+1} = 0$$

then the error goes to zero and *x* is in the open interval of convergence.

A few patterns we now know:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots \quad \Rightarrow \quad e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

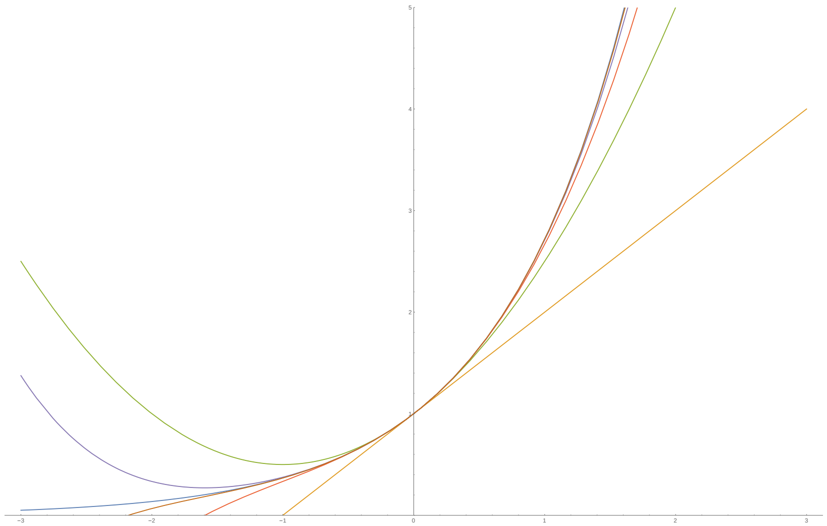
$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \quad \Rightarrow \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$$

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots \quad \Rightarrow \quad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$$

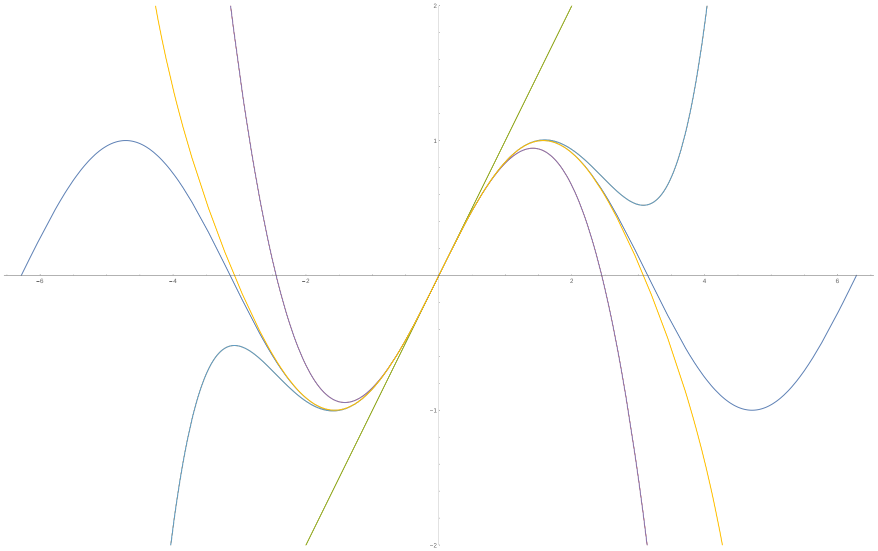
These three converge for ALL values of *x*. So the **open interval of convergence** for each series above is (-∞,∞)

Visuals of Taylor Polynomials:

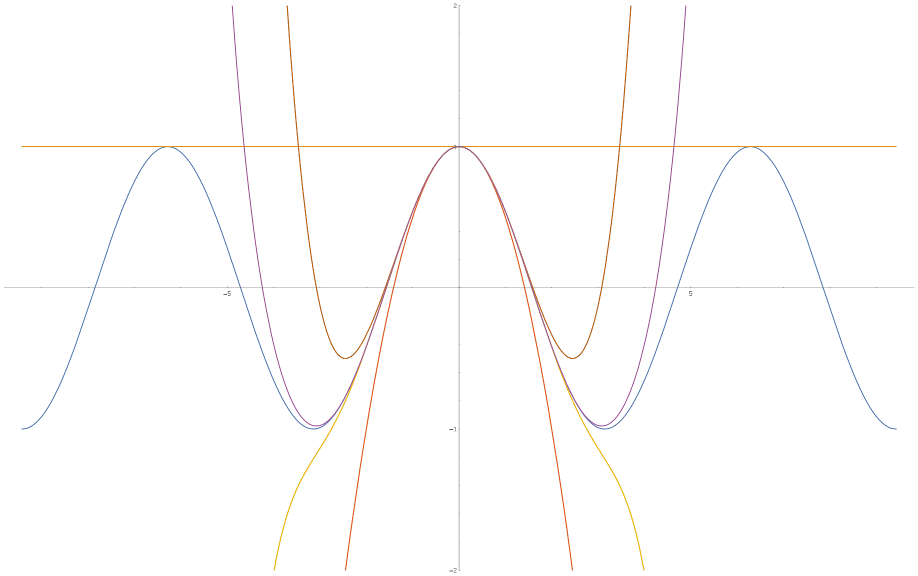
1. $f(x) = e^x$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$ and $T_5(x)$ are shown:



2. $f(x) = \sin(x)$ as well as $T_1(x)$, $T_3(x)$, $T_5(x)$, and $T_7(x)$ are shown:



3. $f(x) = \cos(x)$ as well as $T_1(x)$, $T_2(x)$, $T_4(x)$, $T_6(x)$, and $T_8(x)$ are shown:



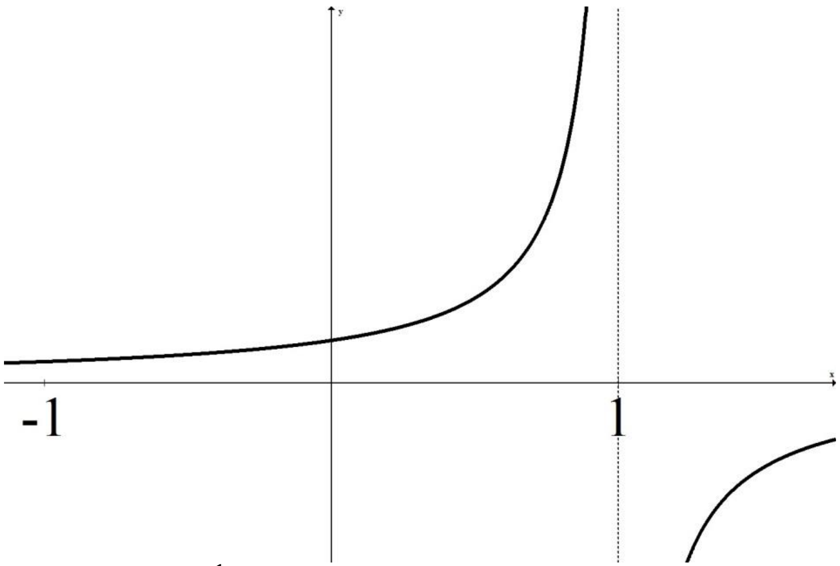
Now consider $f(x) = \frac{1}{1-x}$ based at 0.

Find the 10^{th} Taylor polynomial.

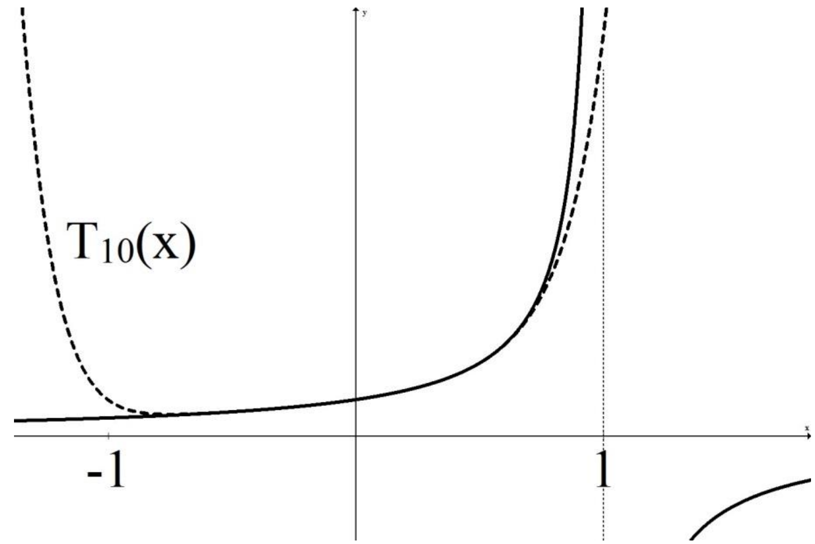
What is the error bound on [-1/2,1/2]?

What is the error bound on [-2,2]?

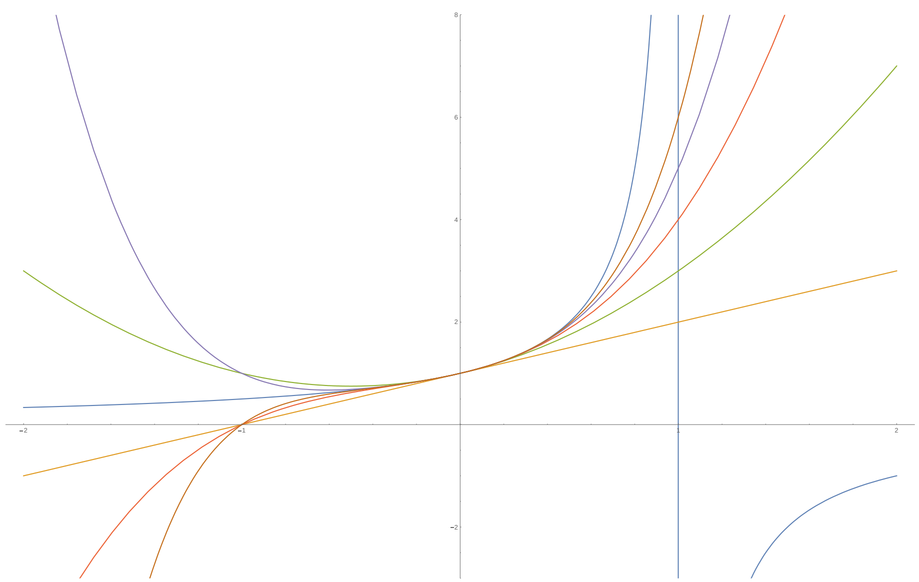
Graph of y = 1/(1-x):



Graph of $f(x) = \frac{1}{1-x}$ and $T_{10}(x)$:



$f(x) = \frac{1}{1-x}$ as well as $T_1(x)$, $T_2(x)$, $T_3(x)$, $T_4(x)$, and $T_5(x)$ are shown:



By Friday, we discuss all the following:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \quad \Rightarrow \quad \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$-\ln(1-x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots \quad \Rightarrow \quad -\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1}$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \cdots \quad \Rightarrow \quad \arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

The open interval of convergence for all three of these series: **-1 < x < 1**.

Sigma Notation Notes

Definition:

$$\sum_{k=a}^{b} f(k) = f(a) + f(a+1) + f(a+2) + \cdots + f(b-1) + f(b)$$

You try: Expand these

$$\sum_{i=1}^{3} \frac{(-1)^i}{i^2} x^i$$

$$\sum_{k=13}^{15} \frac{(-1)^{(k-12)}}{(k-12)^2} x^{k-12}$$

Note: In the examples, above *i* and *k* are *dummy* variables, used to summarize a pattern.

Constants and adding:

Expand then combine

$$5\sum_{k=2}^{4} k^2 x^k - 6\sum_{k=2}^{4} \frac{1}{k!} x^k$$

Summary: For adding/subtracting and constant multiples, you can manipulate in the same way you learned to manipulate integrals.

Derivatives and Integrals

Recall:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C \, , \qquad \frac{d}{dx}(x^n) = n x^{n-1}$$

Thus,

To differentiate a Taylor series → change x^k to $k x^{k-1}$

To integrate a Taylor series → change x^k to $\frac{1}{k+1} \, x^{k+1}$

Example: Find the derivative and general antiderivative of

$$f(x) = -x + \frac{1}{8}x^2 - \frac{1}{27}x^3 + \frac{1}{64}x^4 - \frac{1}{125}x^5 \qquad = \sum_{k=1}^{5} \frac{(-1)^k}{k^3} x^k$$